



10 Mathematics

Theory and Exercises on Factoring and Solving Cubic Polynomials

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Contents

Theory (3-9)

Meaning of the term ‘cubic polynomial’	3
Solving cubic polynomials using ‘factoring by grouping’ ..	4
Solving cubic polynomials using the ‘rational root theorem’ and ‘synthetic division’	9

Exercises (21)

1 - Meaning of the term 'cubic polynomial'

What do we mean by

Cubic¹ Polynomial² ?

1 - 'Cubic' refers to a *cubic function*, where the highest exponent of the *variable* (e.g. x) is 3 - i.e. x^3 .

2 - 'Poly' meaning *many*, and 'nomial' in this case referring to *terms*. So *polynomial* refers to an expression with *many* or more than *one* term.

Therefore, we can say that a *cubic polynomial* is a function with *more than one term* where the *highest exponent of the variable* is 3.

$$\text{e.g. } 2x^3 + 3x^2 + 4x + 5$$

The highest exponent present above is 3, and the expression has more than 1 term.

We can refer generally to the terms in a

cubic polynomial as

$$ax^3 + bx^2 + cx + d$$

2 – Solving cubic polynomials using ‘factoring by grouping’

When asked to solve a cubic polynomial for its variable, for instance

$$\textit{given that } x^3 + 2x^2 - 5x - 10,$$

solve for x,

arguably the most sensible first method that should be tried is that of *factoring by grouping*.

First, let us discuss what is meant by *factoring*.

To *factor* an expression, we (1) first find the *greatest common value* that we can divide all terms in that expression by.

We then (2) divide all terms in the expression by that *greatest common value (GCV)*, then (3) place the simplified expression in *parenthesis*, multiplied by the *greatest common value* :

$$(1) \quad -5x + 10 \{GCV \text{ is } -5\}$$

$$(2) \quad \frac{-5x}{-5} - \frac{10}{-5}$$

$$(3) \quad -5(x + 2)$$

Now, to *factor by grouping* first involves a *test* which we use to identify whether we can actually use this method on our expression or not. Also, we will only discuss how to use *factoring by grouping* to solve *cubic polynomials*.

The factoring by grouping test:

1. *Arrange*

Ensure that the *cubic polynomial* is in the form $ax^3 + bx^2 + cx + d$:

$$\begin{aligned} & -10 + x^3 - 5x + 2x^2 \\ & x^3 + 2x^2 - 5x - 10 \end{aligned}$$

2. *Group*

Mentally split the *cubic polynomial* into two *groups*. $(ax^3 + bx^2)$ is a group, as are $(cx + d)$:

$$\boxed{x^3 + 2x^2} \quad \boxed{-5x - 10}$$

3. *Check coefficient ratios*

Check to see if the *ratio* of the *coefficients* of the two terms in the first group is *equal* to that of the second group. If the ratios are equal, then our test is satisfied, and we can use *factoring by grouping*.

a. A successful *factoring by grouping test*:

$$\begin{array}{l} x^3 + 2x^2 \quad -5x - 10 \\ 1x^3 + 2x^2 \quad -5x - 10 \end{array}$$

	Group 1		Group 2	
Coefficients	1	2	-5	-10
Ratio of coefficients	$\frac{1}{2}$		$\frac{-5}{-10} = \frac{1}{2}$	
Is the ratio the same?	Yes, $\frac{1}{2} = \frac{1}{2}$			

b. An unsuccessful *factoring by grouping test*:

$$\begin{array}{l} x^3 + 3x^2 \quad -5x - 10 \\ 1x^3 + 3x^2 \quad -5x - 10 \end{array}$$

	Group 1		Group 2	
Coefficients	1	3	-5	-10
Ratio of coefficients	$\frac{1}{3}$		$\frac{-5}{-10} = \frac{1}{2}$	
Is the ratio the same?	No, $\frac{1}{3} \neq \frac{1}{2}$			

If our test is successful as it was in (a), we can proceed to use the *factoring by grouping* method to solve the *cubic polynomial*.

However, if our test is unsuccessful as it was in (b), we must use a different method which we shall discuss later.

For now, assuming that our test is successful, the steps to solve a *cubic polynomial* using *factoring by grouping* are

1. Perform the *factoring by grouping* test on the *cubic polynomial*.
2. If successful, find the *greatest common factor* of each term within each of the two groups, and *factorise* each group using that *greatest common factor*:

$$\boxed{x^3 + 2x^2} - 5x - 10$$
$$x^2(x + 2) - 5(x + 2)$$

3. If the test was performed correctly, the expressions in the *parenthesis* for both groups should be the same (as above).
4. Factor out the expression in the parenthesis:

$$x^2(x + 2) - 5(x + 2)$$
$$(x + 2)(x^2 - 5)$$

5. We can now set the expression to *zero* and solve for x :

$$(x + 2)(x^2 - 5) = 0$$

$$\therefore x = 2 \text{ or } \pm \sqrt{5}$$

Question for you, can we solve the *cubic polynomial* $-2x^3 - 3x^2 + 6x + 14$ using *factoring by grouping*? And can you explain why or why not?

What about $-2x^3 - 3x^2 + 6x + 12$?

3 – Solving cubic polynomials using the ‘rational root theorem’ and ‘synthetic division’

As we said before, in our attempt to solve a *cubic polynomial*, arguably it is most sensible to first try *factoring by grouping*. We then said that if we find that our *factoring by grouping* test is unsuccessful, this means that we cannot solve the *cubic polynomial* using *factoring by grouping* and hence must use a different method - which we shall now discuss.

Rational root theorem paired with synthetic division method

Let us apply this method to a different *cubic polynomial* which cannot be solved using *factoring by grouping*:

$$x^3 - 4x^2 + x + 6$$

1. We first find a *rational zero* of the *cubic polynomial* using the *rational root theorem*.
 - i. A *rational* value is one that can be expressed as a fraction, and
 - ii. a *zero* is a value for x which makes our *cubic polynomial* equal 0.
 - iii. Therefore a *rational zero* is a value that can be expressed as a fraction and makes our *cubic polynomial* equal 0,
i.e. $ax^3 + bx^2 + cx + d = 0$
in this case, a rational value that makes $x^3 - 4x^2 + x + 6 = 0$

- a. The *rational root theorem* states that a *rational zero* of a *cubic polynomial* can be expressed as

$$\frac{p}{q}$$

Where p is a *factor* of the *constant term* (d) in our *cubic polynomial*:

$$x^3 - 4x^2 + x + 6$$

- i. +6 is called our *constant term* because it is not being multiplied by our variable (x).
- ii. The factors of +6 are ± 1 , ± 2 , ± 3 and ± 6
- iii. Therefore,

$$p = \pm 1 \text{ or } \pm 2 \text{ or } \pm 3 \text{ or } \pm 6$$

And q is a factor of the *coefficient* of x^3 :

$$x^3 - 4x^2 + x + 6$$

$$1x^3 - 4x^2 + x + 6$$

- i. 1 is our *coefficient* of x^3
- ii. The factors of 1 are ± 1 .
- iii. Therefore,

$$q = \pm 1$$

So, our potential values for p and q are

$$p = \pm 1 \text{ or } \pm 2 \text{ or } \pm 3 \text{ or } \pm 6$$

$$q = \pm 1$$

- b. Now that we have found all of our potential values for p and q , we can substitute any combination of them into the *rational root theorem*, then test to see if our result is a *rational zero* by substituting said result into our *cubic polynomial*:

i. List $\frac{p}{q}$ combinations:

		q	
		+1	-1
p	+1	$\frac{1}{1} = 1$	$\frac{1}{-1} = -1$
	-1	$\frac{-1}{1} = -1$	$\frac{-1}{-1} = 1$
	+2	$\frac{2}{1} = 2$	$\frac{2}{-1} = -2$
	-2	$\frac{-2}{1} = -2$	$\frac{-2}{-1} = 2$
	+3	$\frac{3}{1} = 3$	$\frac{3}{-1} = -3$
	-3	$\frac{-3}{1} = -3$	$\frac{-3}{-1} = 3$
	+6	$\frac{6}{1} = 6$	$\frac{6}{-1} = -6$
	-6	$\frac{-6}{1} = -6$	$\frac{-6}{-1} = 6$

So, the $\frac{p}{q}$ values that we can try are $\frac{p}{q} = \pm 1, \pm 2, \pm 3, \pm 6$

ii. Substitute individual $\frac{p}{q}$ combinations into the *cubic polynomial*

until a $\frac{p}{q}$ value is found that makes $ax^3 + bx^2 + cx + d = 0$

(let's start with +1, then -1, then +2, then -2 and so forth ...)

$\frac{p}{q}$	Substituting into $x^3 - 4x^2 + x + 6$	Is this $\frac{p}{q}$ value a zero?
1	$(1)^3 - 4(1)^2 + (1) + 6 = 4$	No
-1	$(-1)^3 - 4(-1)^2 + (-1) + 6 = 0$	Yes

(we were lucky to find a rational zero on our second attempt!)

- c. Once we have found a rational zero for our *cubic polynomial* using *rational root theorem*, which we will call z , in our case,

$$z = -1 \text{ because}$$

$x = -1$ is a rational zero of

$$x^3 - 4x^2 + x + 6$$

We know that $(x - z)$, or in our case $(x - (-1)) = (x + 1)$, will be a *factor* of our *cubic polynomial* - meaning that we could convert our cubic polynomial

$$ax^3 + bx^2 + cx + d$$

into the form

$$(x - z)(ex^2 + fx + g)$$

or in our case

$$(x + 1)(ex^2 + fx + g)$$

which is much easier to solve!

However, to find $(ex^2 + fx + g)$, we will need to

divide our cubic polynomial
by $(x - z)$

Since $(x - z) \times (ex^2 + fx + g) = ax^3 + bx^2 + cx + d$

So in our case we are trying to

divide $x^3 - 4x^2 + x + 6$
by $(x + 1)$

i. e.
$$\frac{ax^3 + bx^2 + cx + d}{x - z}$$

in our case
$$\frac{x^3 - 4x^2 + x + 6}{x + 1}$$

And we can do this very quickly using

synthetic division

which has the following steps:

Synthetically dividing $ax^3 + bx^2 + cx + d$ by $(x - z)$:

1. Draw the following structure:



2. Write the zero or z that we found using *rational root theorem* here:

$$\frac{x^3 - 4x^2 + x + 6}{x + 1}$$



$$\frac{ax^3 + bx^2 + cx + d}{x - z}$$

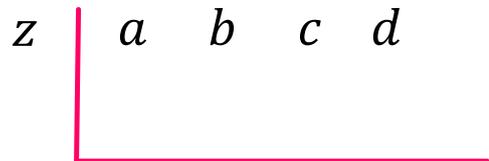


3. Write out the coefficients of the *cubic polynomial* in a, b, c, d order:

$$\frac{x^3 - 4x^2 + x - 6}{x + 1}$$



$$\frac{ax^3 + bx^2 + cx + d}{x - z}$$



4. Bring down the a coefficient:

$$\frac{x^3 - 4x^2 + x + 6}{x + 1} \quad \downarrow$$

$$\frac{ax^3 + bx^2 + cx + d}{x - z} \quad \downarrow$$

-1	1	-4	1	6
	1			

z	a	b	c	d
	a			

5. Multiply the number brought down, by z , and write it here:

$$\frac{x^3 - 4x^2 + x + 6}{x + 1} \quad \downarrow$$

$$\frac{ax^3 + bx^2 + cx + d}{x - z} \quad \downarrow$$

-1	1	-4	1	6
		-1		
	1			

z	a	b	c	d
	az			
	a			

8. The very last number brought down will be our remainder, which we express as $\frac{\text{remainder}}{(x-z)}$, but in this case it is conveniently 0.

$$\frac{x^3 - 4x^2 + x + 6}{x + 1}$$

↓

$$\begin{array}{r|rrrr} -1 & 1 & -4 & 1 & 6 \\ & & -1 & 5 & -6 \\ \hline & 1 & -5 & 6 & 0 \end{array}$$

9. What is fascinating about this method, is that the numbers that we have brought down are the coefficients of $ex^2 + fx + g$! Plus, a remainder if there is one

$$\frac{x^3 - 4x^2 + x + 6}{x + 1}$$

↓

$$\begin{array}{r|rrrr} -1 & 1 & -4 & 1 & 6 \\ & & -1 & 5 & -6 \\ \hline & 1 & -5 & 6 & 0 \\ & \downarrow & \downarrow & \downarrow & \downarrow \\ & 1x^2 & -5x & +6 & +0 \end{array}$$

10. Now that we have found $ex^2 + fx + g$ using synthetic division, and our *zero* using *rational root theorem*, we can rewrite our cubic polynomial $ax^3 + bx^2 + cx + d$ as

$$\begin{aligned} & ax^3 + bx^2 + cx + d \\ &= (x - z)(ex^2 + fx + g) \end{aligned}$$

In our case:

$$\begin{aligned} & x^3 - 4x^2 + x - 6 \\ &= (x + 1)(1x^2 - 5x + 6 + 0) \\ &= (x + 1)(x^2 - 5x + 6) \end{aligned}$$

And we can further factorise the $ex^2 + fx + g$ expression if needed:

$$\begin{aligned} & (x + 1)(x^2 - 5x + 6) \\ &= (x + 1)(x - 2)(x + 3) \end{aligned}$$

And finally, solve our *cubic polynomial*:

$$\therefore x = -3, -1, \text{ or } 2$$

To revise, we discussed two methods for solving *cubic polynomials* of the form

$$ax^3 + bx^2 + cx + d:$$

1. Perform the *factoring by grouping test*. If successful, *factor by grouping*.
2. If the *factoring by grouping test* is unsuccessful,
 - i. use the *rational root theorem* to find a zero of the *cubic polynomial*, expressed as z
 - ii. Know that $(x - z)$ will divide into $ax^3 + bx^2 + cx + d$ to produce $ex^2 + fx + g$.
 - iii. Find $ex^2 + fx + g$ using synthetic division.
 - iv. Rewrite $ax^3 + bx^2 + cx + d$ as $(x - z)(ex^2 + fx + g)$
 - v. Further factorise $(ex^2 + fx + g)$ if needed
 - vi. Set the expression to 0 and solve for x

4 – Exercises

1. Solve the following *cubic polynomials*

a. $x^3 - 2x^2 - 5x + 6$

b. $x^3 + 2x^2 - 16x - 32$

c. $3x^3 - 2x^2 - 150$

d. $n^3 + 5n^2 - 9n - 45$