Proving the irrationality of $\sqrt{3}$ using a proof of divisibility

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1 – Problem posed

(1) Prove that if an integer n is divisible by 3, then n^2 is also divisible by 3.

(2) Hence, prove that $\sqrt{3}$ is irrational.

2 – Proposed solution

(1) Prove that if an integer n is divisible by an 3, then n^2 is also divisible by 3.

The stem has granted us that 'an integer n is divisible by 3'.

Therefore,

$$n = 3k$$

Because, if n is divisible by 3, this implies that n is a *multiple* of 3.

Now, we want to prove that n^2 is also divisible by 3, so let us make n^2 the subject of the equation:

$$n = 3k$$

$$n^2 = 3^2k^2$$

To prove that n^2 is also divisible by 3, we must show that it is also a multiple of 3.

I.e. that $n^2 = 3 \times$ an integer. We can factor out a 3 from 3^2 to do this:

$$n^2 = 3^2k^2$$

$$n^2 = 3(3k^2)$$

We have now proved the first leg of the problem.

(2) Hence, prove that $\sqrt{3}$ *is* irrational.

There are two ways that we could do this:

1. Show that $\sqrt{3}$ is irrational directly
2. Show that $\sqrt{3}$ is *not rational*, thereby indirectly showing that $\sqrt{3}$ must be irrational ('proving by contradiction')

We shall use the second method:

If $\sqrt{3}$ is a rational number, we should be able to write it in the form $\frac{p}{q}$ where:

1. p and q are not 0 (because $\sqrt{3} \neq 0$, and we cannot divide by 0)
2. p and q are *coprime*, meaning that the only common factor that they share is 1,
3. and $\frac{p}{q}$ is in lowest terms (as simplified as possible)

Since we are proving by contradiction, let us assume that $\sqrt{3}$ is rational, so

$$\sqrt{3} = \frac{p}{q} \;\; \{p, q \neq 0 \text{ and } \gcd(p, q) = 1\}$$

Now, if we can prove that p and q are *not* coprime, meaning that p and q share a factor that is greater than 1, then we can prove that $\sqrt{3}$ is not rational:

$$3 = \frac{p^2}{q^2}$$

$$p^2 = 3q^2$$

$$\therefore p^2 \text{ is divisible by } 3$$

$$\therefore p \text{ is divisible by } 3$$

Because we proved earlier that if a number is divisible by 3, then its square is also divisible by three – and this is true vice-versa. This also means that 3 is a divisor of p.

Now, if we can prove that 3 is also a factor of q, we will have proved that p and q share a factor of 3, meaning that they are not coprime, meaning that $\sqrt{3}$ is not rational.

$$p^2 = 3q^2$$

$$\therefore q^2 = 3k^2 \quad \{\text{From leg 1 we found that } n^2 = 3(3k^2)\}$$

$$\therefore q^2 \text{ is divisible by } 3$$

$$\therefore q \text{ is divisible by } 3$$

$$\therefore p \text{ and } q \text{ share a common factor of } 3$$

But coprime numbers only share a common factor of 1.

$$\therefore p \text{ and } q \text{ are not coprime}$$

$$\therefore \sqrt{3} \text{ is irrational}$$