



11 Mathematics

Proposed solution to a problem involving
quadratic identities, summation
notation, polynomial sequence
differences, and binomial expansion

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1 – Problem posed

- a. Find constants a , b , and c such that

$$x^2 = a(x - 1)^2 + b(x - 3)^2 + c(x - 4)^2$$

- b. Hence, use your result to evaluate

$$1^2 + 2^2 + 3^2 + \dots + 20^2$$

by expressing x^2 in terms of shifted squares.

2 – Proposed solution to (a)

a. Find constants a , b , and c such that

$$x^2 = a(x-1)^2 + b(x-3)^2 + c(x-4)^2$$

(a) $x^2 = a(x-1)^2 + b(x-3)^2 + c(x-4)^2$

Let $x=1$,

$$a(0)^2 + b(-2)^2 + c(-3)^2 = (1)^2$$
$$4b + 9c = 1 \quad (1)$$

Let $x = 3$

$$a(2)^2 + b(0)^2 + c(-1)^2 = (3)^2$$
$$4a + c = 9$$
$$c = 9 - 4a \quad (2)$$

Let $x = 4$

$$a(3)^2 + b(1)^2 + c(0)^2 = (4)^2$$
$$9a + b = 16$$
$$b = 16 - 9a \quad (3)$$

Notes:

Substituting $x = 1$, $x = 3$ and $x = 4$ to create equations to be solved simultaneously for a , b , and c .

(2) into (1),

$$4b + 9(9 - 4a) = 1 \quad \{(2)\}$$

$$4b + 81 - 36a = 1 \quad (1A)$$

(3) into (1A),

$$4(16 - 9a) + 81 - 36a = 1 \quad \{(3)\}$$

$$64 + 81 - 36a - 36a = 1$$

$$-72a = -144$$

$$a = 2 \quad (1Ai)$$

(1Ai) into (2),

$$c = 9 - 4(2) \quad \{(1Ai)\}$$

$$c = 1 \quad (2A)$$

Notes:

Solving simultaneously for a and c .

(2A) into (1),

$$4b + 9(1) = 1 \quad \{ (2A) \}$$

$$4b = -8$$

$$b = -2 \quad (1B)$$

$$\therefore x^2 = 2(x-1)^2 - 2(x-3)^2 + (x-4)^2$$

$$\{ (1Ai), (1B), (2A) \}$$

Notes:

- Solving simultaneously for b .
- Substituting a , b and c back into the original equation to complete Part (a).

3 - Proposed solution to (b)

b. Hence, use your result to evaluate

$$1^2 + 2^2 + 3^2 + \dots + 20^2$$

by expressing x^2 in terms of shifted squares.

(b)

$$x^2 = 2(x-1)^2 - 2(x-3)^2 + (x-4)^2$$
$$\sum_{x=1}^n x^2 = \sum_{x=1}^n [2(x-1)^2 - 2(x-3)^2 + (x-4)^2]$$

1. The problem is requesting the sum of the first 20 square numbers ($1^2 + 2^2 + 3^2 + \dots + 20^2$).

We first express the sum of the first n square numbers more concisely by applying summation notation to x^2 and the equation we were asked to determine in (a).

$$\sum_{x=1}^n x^2 = \sum_{x=1}^n [2(x-1)^2 - 2(x-3)^2 + (x-4)^2]$$
$$= 2 \sum_{x=1}^n (x-1)^2 - 2 \sum_{x=1}^n (x-3)^2 + \sum_{x=1}^n (x-4)^2$$

2. Expanding the summation to our individual terms

$$\begin{aligned}
\sum_{x=1}^n x^2 &= \sum_{x=1}^n [2(x-1)^2 - 2(x-3)^2 + (x-4)^2] \\
&= 2 \sum_{x=1}^n (x-1)^2 - 2 \sum_{x=1}^n (x-3)^2 + \sum_{x=1}^n (x-4)^2 \\
&= 2 \sum_{x=1-1}^{n-1} [(x+1)-1]^2 - 2 \sum_{x=1-3}^{n-3} [(x+3)-3]^2 + \sum_{x=1-4}^{n-4} [(x+4)-4]^2 \\
&= 2 \sum_{x=0}^{n-1} x^2 - 2 \sum_{x=-2}^{n-3} x^2 + \sum_{x=-3}^{n-4} x^2
\end{aligned}$$

3. Shifting the indices of the summations as requested by the stem, so that each summation is expressed in terms of x^2 .

$$\begin{aligned}
\sum_{x=1}^n x^2 &= \sum_{x=1}^n [2(x-1)^2 - 2(x-3)^2 + (x-4)^2] \\
&= 2 \sum_{x=1}^n (x-1)^2 - 2 \sum_{x=1}^n (x-3)^2 + \sum_{x=1}^n (x-4)^2 \\
&= 2 \sum_{x=1-1}^{n-1} [(x+1)-1]^2 - 2 \sum_{x=1-3}^{n-3} [(x+3)-3]^2 + \sum_{x=1-4}^{n-4} [(x+4)-4]^2 \\
&= 2 \sum_{x=0}^{n-1} x^2 - 2 \sum_{x=-2}^{n-3} x^2 + \sum_{x=-3}^{n-4} x^2 \\
&= 2[(0)^2 + \dots + (n-1)^2] - 2[(-2)^2 + (-1)^2 + \dots + (n-3)^2] \\
&\quad + [(-3)^2 + \dots + (n-4)^2]
\end{aligned}$$

4. Writing out the partial results of the summations in order to telescope (determine which terms will cancel without writing out the entire results of the summations).

$$\sum_{x=1}^n x^2 = \sum_{x=1}^n [2(x-1)^2 - 2(x-3)^2 + (x-4)^2]$$

$$= 2 \sum_{x=1}^n (x-1)^2 - 2 \sum_{x=1}^n (x-3)^2 + \sum_{x=1}^n (x-4)^2$$

$$= 2 \sum_{x=1-1}^{n-1} [(x+1)-1]^2 - 2 \sum_{x=1-3}^{n-3} [(x+3)-3]^2 + \sum_{x=1-4}^{n-4} [(x+4)-4]^2$$

$$= 2 \sum_{x=0}^{n-1} x^2 - 2 \sum_{x=-2}^{n-3} x^2 + \sum_{x=-3}^{n-4} x^2$$

$$= 2[(0)^2 + \dots + (n-1)^2] - 2[(-2)^2 + (-1)^2 + \dots + (n-3)^2] + [(-3)^2 + \dots + (n-4)^2]$$

$$= 2(n-2)^2 + 2(n-1)^2 - (-2)^2 - (-1)^2 + [(0)^2 + \dots + (n-4)^2]$$

5. One set of $[(0)^2 + \dots + (n-3)^2]$ cancels, and we write the terms that remain.

$$\sum_{x=1}^n x^2 = \sum_{x=1}^n [2(x-1)^2 - 2(x-3)^2 + (x-4)^2]$$

$$= 2 \sum_{x=1}^n (x-1)^2 - 2 \sum_{x=1}^n (x-3)^2 + \sum_{x=1}^n (x-4)^2$$

$$= 2 \sum_{x=1-1}^{n-1} [(x+1)-1]^2 - 2 \sum_{x=1-3}^{n-3} [(x+3)-3]^2 + \sum_{x=1-4}^{n-4} [(x+4)-4]^2$$

$$= 2 \sum_{x=0}^{n-1} x^2 - 2 \sum_{x=-2}^{n-3} x^2 + \sum_{x=-3}^{n-4} x^2$$

$$= 2[(0)^2 + \dots + (n-1)^2] - 2[(-2)^2 + (-1)^2 + \dots + (n-3)^2] + [(-3)^2 + \dots + (n-4)^2]$$

$$= 2(n-2)^2 + 2(n-1)^2 - (-2)^2 - (-1)^2 + [(0)^2 + \dots + (n-4)^2]$$

$$= 2(n^2 - 4n + 4) + 2(n^2 - 2n + 1) - 4 - 1 + 9 + \sum_{x=1}^{n-4} x^2$$

$$S_n = 2n^2 - 8n + 8 + 2n^2 - 4n + 2 + 4 + S_{n-4}$$

$$= 4n^2 - 12n + 14 + S_{n-4}$$

6. Expanding and collecting like terms.

7. Expressing $\sum_{x=1}^n x^2$ as S_n and $\sum_{x=1}^{n-4} x^2$ as S_{n-4} for concision.

$$\begin{aligned}
\sum_{x=1}^n x^2 &= \sum_{x=1}^n [2(x-1)^2 - 2(x-3)^2 + (x-4)^2] \\
&= 2 \sum_{x=1}^n (x-1)^2 - 2 \sum_{x=1}^n (x-3)^2 + \sum_{x=1}^n (x-4)^2 \\
&= 2 \sum_{x=1-1}^{n-1} [(x+1)-1]^2 - 2 \sum_{x=1-3}^{n-3} [(x+3)-3]^2 + \sum_{x=1-4}^{n-4} [(x+4)-4]^2 \\
&= 2 \sum_{x=0}^{n-1} x^2 - 2 \sum_{x=-2}^{n-3} x^2 + \sum_{x=-3}^{n-4} x^2 \\
&= 2[(0)^2 + \dots + (n-1)^2] - 2[(-2)^2 + (-1)^2 + \dots + (n-3)^2] \\
&\quad + [(-3)^2 + \dots + (n-4)^2] \\
&= 2(n-2)^2 + 2(n-1)^2 - (-2)^2 - (-1)^2 + [(0)^2 + \dots + (n-4)^2] \\
&= 2(n^2 - 4n + 4) + 2(n^2 - 2n + 1) - 4 - 1 + 9 + \sum_{x=1}^{n-4} x^2 \\
S_n &= 2n^2 - 8n + 8 + 2n^2 - 4n + 2 + 4 + S_{n-4} \\
&= 4n^2 - 12n + 14 + S_{n-4} \\
4n^2 - 12n + 14 &= S_n - S_{n-4}
\end{aligned}$$

8. Rearranging to show that $\Delta(S_n - S_{n-4})$ equals a quadratic sequence.

$\Delta(S_n - S_{n-4})$ represents the 'rate of growth' between two points on the function S_n . (If we let $n = 1$, $S_n = S_1$, which would represent the point on a graph $(1, S_1)$, $S_{n-4} = S_{-3}$ which would represent the point of a graph $(-3, S_{-3})$. If the rate of growth of a function between two points can be expressed as a quadratic function ($4n^2 - 12n + 14$ in our case), then the original function (S_n) is a cubic function.

$$4n^2 - 12n + 14 = S_n - S_{n-4}$$

$$= (an^3 + bn^2 + cn + d) - [a(n-4)^3 + b(n-4)^2 + c(n-4) + d] \quad \{\Delta(S_n - S_{n-4}) \text{ is quadratic}\}$$

9. Since we know that S_n is a cubic function, we know that it and S_{n-4} can be expressed as above.

$$4n^2 - 12n + 14 = S_n - S_{n-4}$$

$$= (an^3 + bn^2 + cn + d) - [a(n-4)^3 + b(n-4)^2 + c(n-4) + d] \quad \{\Delta(S_n - S_{n-4}) \text{ is quadratic}\}$$

$$= an^3 + bn^2 + cn + d - [(an^3 - 12an^2 + 48an - 64a) + (bn^2 - 8bn + 16b) + (cn - 4c) + d]$$

$$= 12an^2 - (48a - 8b)n + (64a - 16b + 4c)$$

10. Expanding and collecting like terms, as well as attempting to express in the format $an^2 + bn + c$ to compare coefficients with $4n^2 - 12n + 14$.

$$4n^2 - 12n + 14 = S_n - S_{n-4}$$

$$= (an^3 + bn^2 + cn + d) - [a(n-4)^3 + b(n-4)^2 + c(n-4) + d] \quad \{\Delta(S_n - S_{n-4}) \text{ is quadratic}\}$$

$$= an^3 + bn^2 + cn + d - [(an^3 - 12an^2 + 48an - 64a) + (bn^2 - 8bn + 16b) + (cn - 4c) + d]$$

$$= 12an^2 - (48a - 8b)n + (64a - 16b + 4c)$$

$$= 4n^2 - (16 - 8b)n + \left(\frac{64}{3} - 16b + 4c\right)$$

$$\left\{ \begin{array}{l} \text{Comparing coefficients} \\ (1) \end{array} \right. \quad \left. \begin{array}{l} 4n^2 = 12an^2 \\ \therefore a = \frac{1}{3} \end{array} \right\}$$

$$= 4n^2 - 12n + \left(\frac{40}{3} + 4c\right)$$

$$\left\{ \begin{array}{l} \text{Comparing coefficients} \\ (2) \end{array} \right. \quad \left. \begin{array}{l} 16 - 8b = 12 \\ \therefore b = \frac{1}{2} \end{array} \right\}$$

$$= 4n^2 - 12n + 14$$

$$\left\{ \begin{array}{l} \text{Comparing coefficients} \\ (3) \end{array} \right. \quad \left. \begin{array}{l} \frac{40}{3} + 4c = 14 \\ \therefore c = \frac{1}{6} \end{array} \right\}$$

$$\therefore S_n = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n \quad \{(1), (2), (3)\}$$

$$S_n = \frac{2n^3 + 3n^2 + n}{6}$$

12. Substituting a , b and c into S_n and expressing the terms using their lowest common denominator.

$$S_n = \frac{n(2n^2 + 3n + 1)}{6}$$

$$S_n = \frac{n(2n+1)(n+1)}{6}$$

$$\left. \begin{aligned} \text{When } S_n &= 0, \\ n &= \frac{-3 \pm \sqrt{3^2 - 4(2)(1)}}{2(2)} \\ &= \frac{-3 \pm \sqrt{1}}{4} \\ \therefore n &= -\frac{1}{2}, 1 \end{aligned} \right\}$$

13. Factorising n

14. Using the quadratic formula to find the zeroes of and factorise $2n^2 + 3n + 1$, and derive the known formula for the sum of the squares of the first n natural numbers.

$$S_n = \frac{n(2n^2 + 3n + 1)}{6}$$

$$S_n = \frac{n(2n+1)(n+1)}{6}$$

$$\left\{ \begin{array}{l} \text{When } S_n = 0, \\ n = \frac{-3 \pm \sqrt{3^2 - 4(2)(1)}}{2(2)} \\ = \frac{-3 \pm \sqrt{1}}{4} \\ \therefore n = -\frac{1}{2}, 1 \end{array} \right.$$

$$\therefore 1^2 + 2^2 + 3^2 + \dots + 20^2 = \sum_{x=1}^{20} x^2$$

$$= S_{20}$$

$$= \frac{20(2(20)+1)(20+1)}{6}$$

$$= 2870$$

15. Using the derived formula to evaluate $1^2 + 2^2 + 3^2 + \dots + 20^2$