Theory on solving a trigonometric equation

Tolentino Tuition
Modbury Heights, tolentinotuition.com
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Problem and explanation

PROBLEM

Solve the following equation for x on $0 \leq x \leq 2\pi$:

$$\tan^2 3x = \frac{1}{3}$$

EXPLANATION

1. $\tan^2 3x$ can be rewritten as $(\tan 3x)^2$

$$\tan^2 3x = \frac{1}{3}$$

$$(\tan 3x)^2 = \frac{1}{3}$$

2. We can now take the square root of both sides to remove the square.

   But we must write $\pm$ in front of $\sqrt{\frac{1}{3}}$.

$$\sqrt{(\tan 3x)^2} = \pm\sqrt{\frac{1}{3}}$$

This is because you could get $\frac{1}{3}$ by squaring

i. $+\sqrt{\frac{1}{3}}$, because $\left(+\sqrt{\frac{1}{3}}\right)^2 = \sqrt{\frac{1}{3}} \times \sqrt{\frac{1}{3}} = \frac{1}{3}$; or

ii. $-\sqrt{\frac{1}{3}}$, because $\left(-\sqrt{\frac{1}{3}}\right)^2 = -\sqrt{\frac{1}{3}} \times -\sqrt{\frac{1}{3}} = \frac{1}{3}$

(a negative number $\times$ a negative number is a positive number)

This is why we write $\pm$ when we take the square root of $\frac{1}{3}$, because technically it has two square roots, $+\sqrt{\frac{1}{3}}$ or $-\sqrt{\frac{1}{3}}$

$$\sqrt{(\tan 3x)^2} = \pm\sqrt{\frac{1}{3}}$$

3. We can now simplify the left-hand side, and rewrite $\sqrt{\frac{1}{3}}$ as $\frac{\sqrt{1}}{\sqrt{3}}$, because $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$. And since $\sqrt{1} = 1$, $\frac{\sqrt{1}}{\sqrt{3}} = \frac{1}{\sqrt{3}}$.

$$\tan 3x = \pm\frac{\sqrt{1}}{\sqrt{3}} = \pm\frac{1}{\sqrt{3}}$$

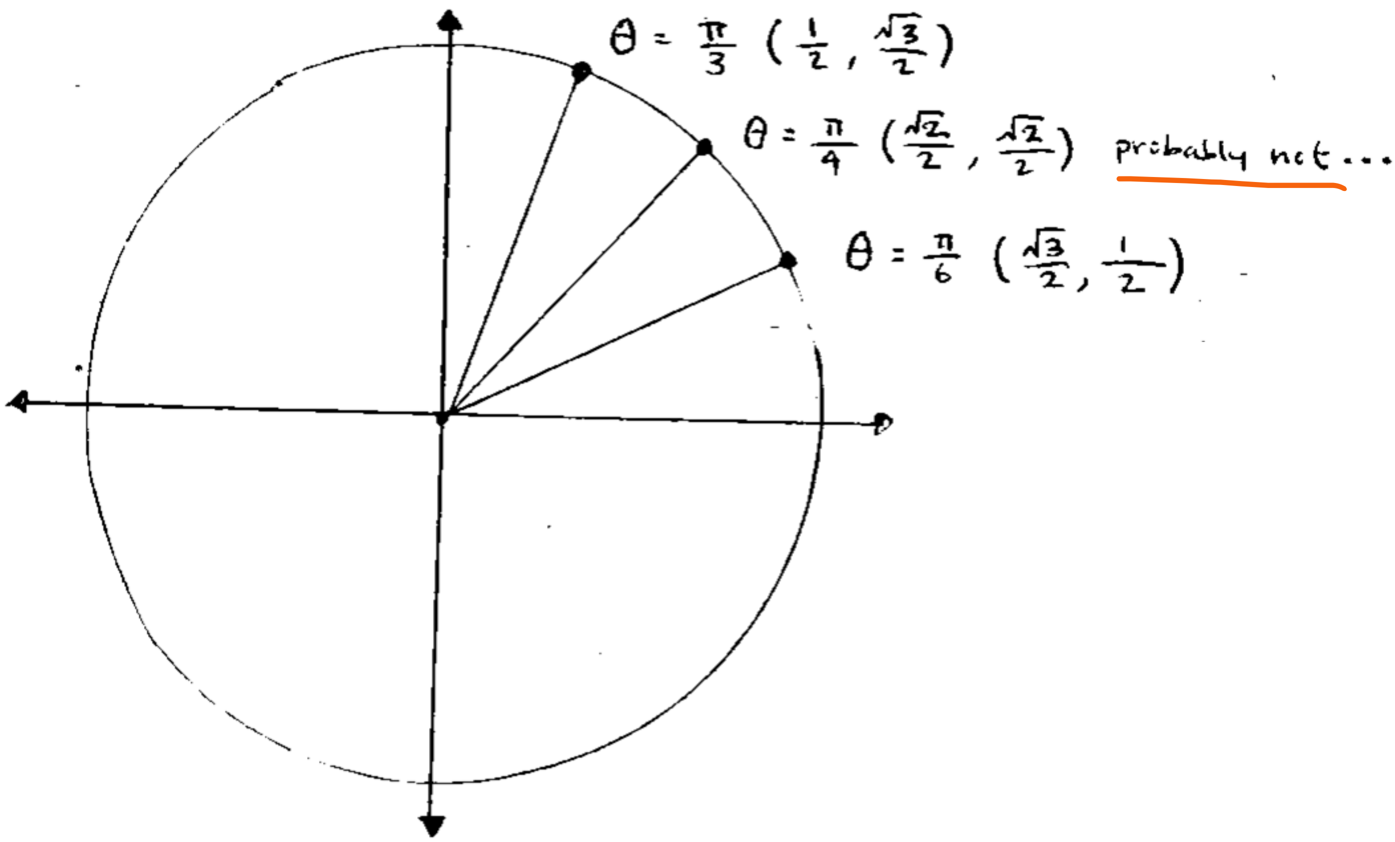
4. $\tan 3x = \pm\frac{1}{\sqrt{3}}$ means that the value of *tan* at some angle '$3x$' will equal $+\frac{1}{\sqrt{3}}$ or $-\frac{1}{\sqrt{3}}$.

But how do we determine the value of *tan* at some angle?

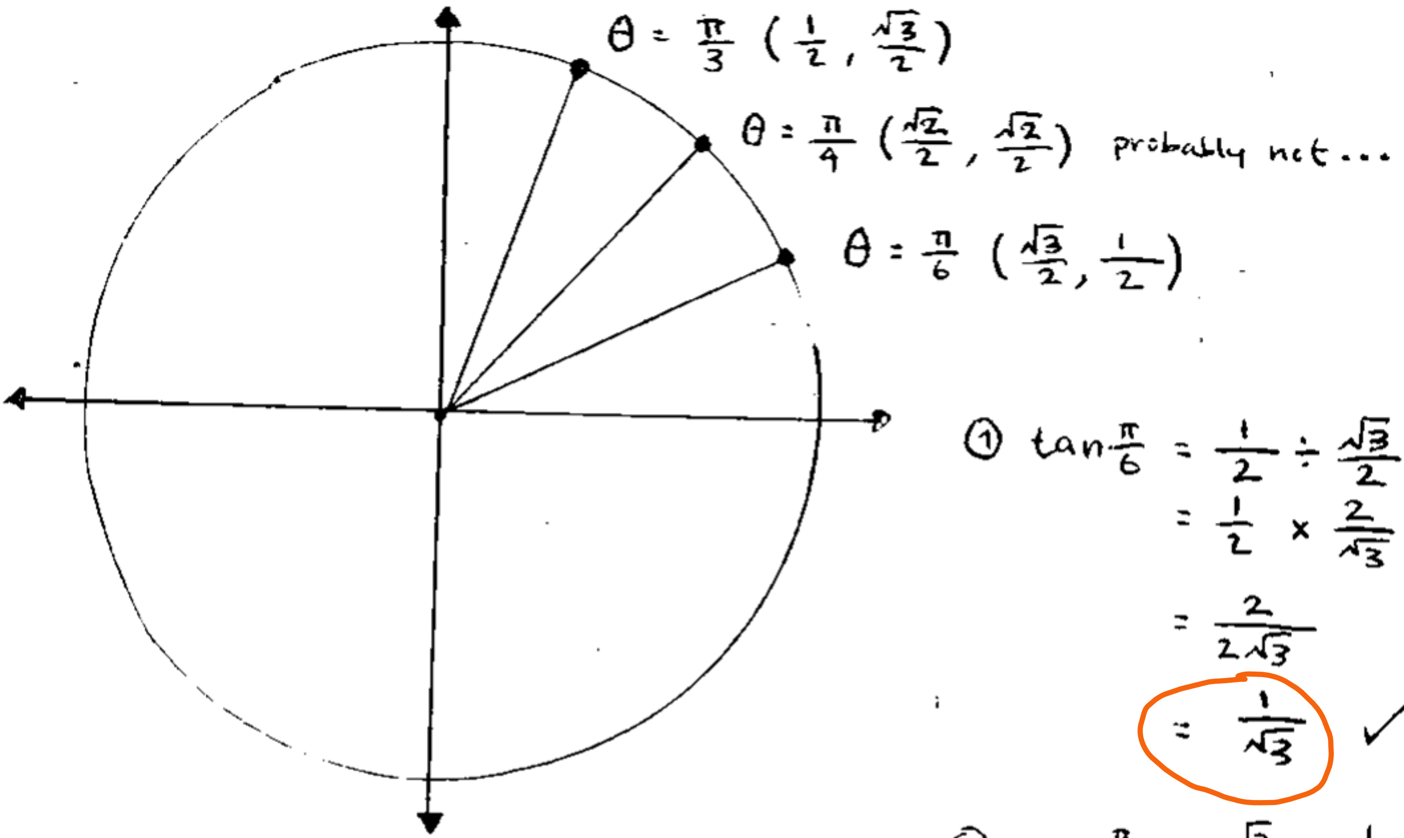
$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Meaning, the value of *tan* at some angle on the unit circle equals the y coordinate of the point at that angle ($\sin\theta$), divided by the x coordinate of the point at that angle ($\cos\theta$).

So, in order to remove *tan* from our equation, we need to find an angle on the unit circle, where the y coordinate divided by the x coordinate will give us either $+\frac{1}{\sqrt{3}}$, or $-\frac{1}{\sqrt{3}}$ (because $\tan 3x = \pm\frac{1}{\sqrt{3}}$, and $3x$ is just some angle).



It definitely won't be $\frac{\pi}{4}$, because the coordinates there are $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, and our *tan* value has a $\sqrt{3}$ in it...
That leaves $\frac{\pi}{6}$ *and* $\frac{\pi}{3}$, and finding the value of *tan* at these angles using $\tan\theta = \frac{\sin\theta}{\cos\theta}$, we find that $tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$, so $\frac{\pi}{6}$ is the angle that we are looking for!



① $\tan\frac{\pi}{6} = \frac{1}{2} \div \frac{\sqrt{3}}{2}$

$= \frac{1}{2} \times \frac{2}{\sqrt{3}}$

$= \frac{2}{2\sqrt{3}}$

$= \frac{1}{\sqrt{3}}$ ✓

② $\tan\frac{\pi}{3} = \frac{\sqrt{3}}{2} \div \frac{1}{2}$

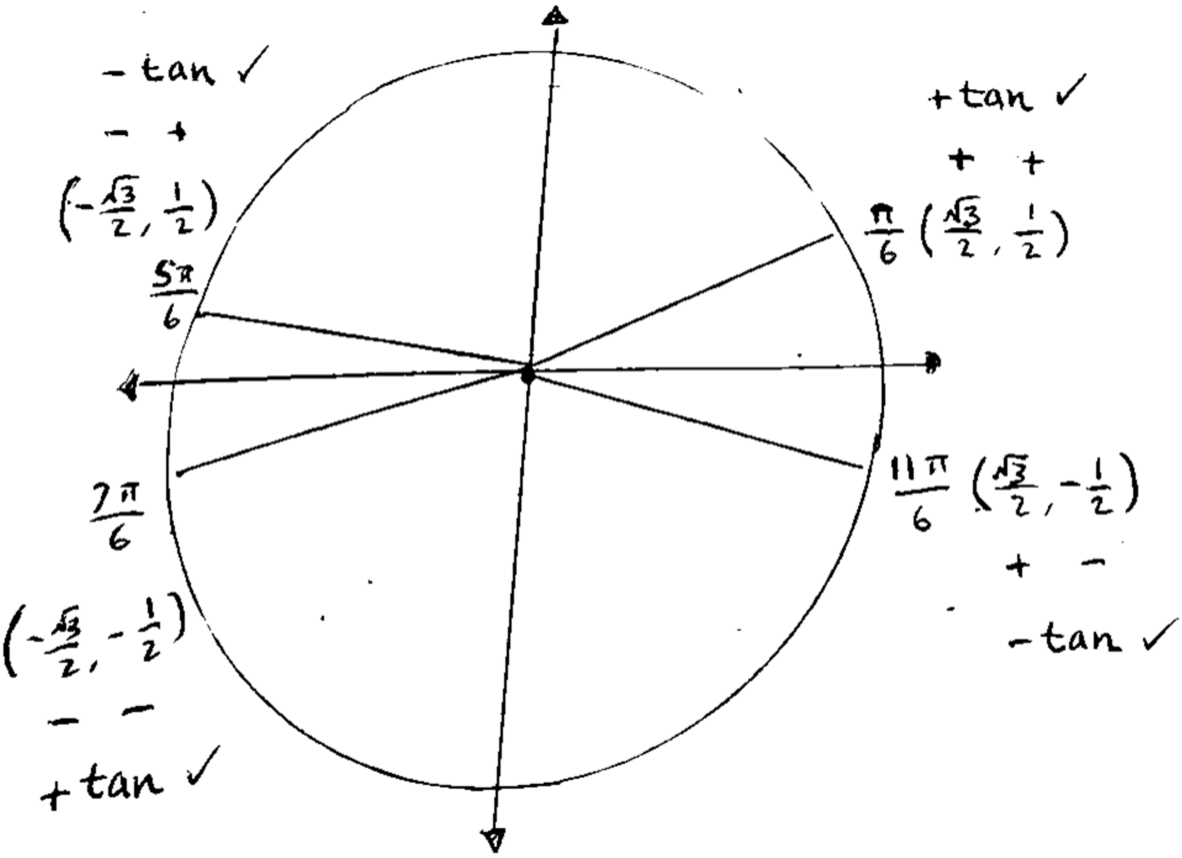
$= \frac{\sqrt{3}}{2} \times \frac{2}{1}$

$= \frac{2\sqrt{3}}{2}$

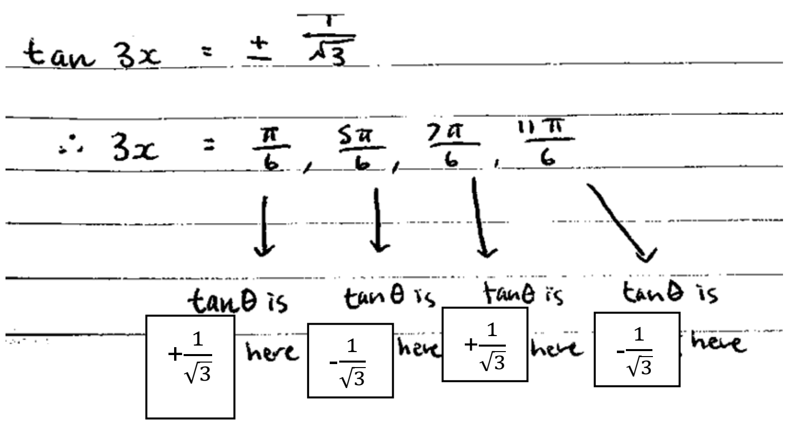
$= \sqrt{3}$ ✗

But we must not forget that $\tan 3x = +OR - \frac{1}{\sqrt{3}}$, so we need to look at the other multiples of $\frac{\pi}{6}$ on the unit circle, between $0 \leq x \leq 2\pi$.

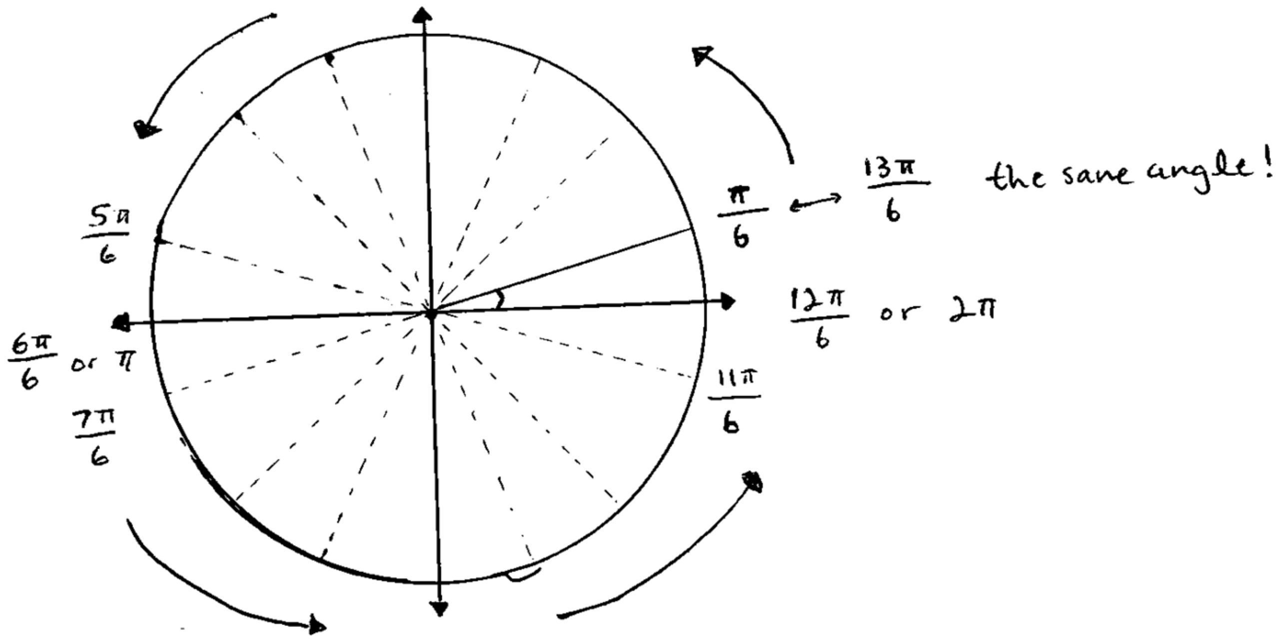
The absolute value of tan at these angles will still be $\frac{1}{\sqrt{3}}$, because the coordinates will always be $(\pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2})$, but tan will either be positive or negative, depending on the quadrant.



Since $\tan 3x = \pm\frac{1}{\sqrt{3}}$, $3x$ could be all of these angles, meaning:



5. But now, are these the *only* angles for which $\tan 3x = \pm\frac{1}{\sqrt{3}}$?
For example, isn't it true that $\tan\frac{\pi}{6}$ and $\tan\frac{13\pi}{6}$ represent the same point on the unit circle? Just a distance of 2π apart!



Since 2π can be expressed as $\frac{12\pi}{6}$, $\frac{\pi}{6}$ and $\frac{13\pi}{6}$ are a distance of 2π apart!
This means that the coordinates at $\tan\frac{13\pi}{6}$ would also be $(\frac{\sqrt{3}}{2},\frac{1}{2})$, so $\tan\frac{13\pi}{6}$ would also equal $\frac{1}{\sqrt{3}}$.

So, if we add 2π (aka. $\frac{12\pi}{6}$) to all of our solutions, $\tan 3x$ would still equal $\pm\frac{1}{\sqrt{3}}$ at all of these angles:

$$\tan 3x = \pm\frac{1}{\sqrt{3}}$$

$$\therefore 3x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \left(\frac{\pi}{6}+\frac{12\pi}{6}\right), \left(\frac{5\pi}{6}+\frac{12\pi}{6}\right), \left(\frac{7\pi}{6}+\frac{12\pi}{6}\right), \left(\frac{11\pi}{6}+\frac{12\pi}{6}\right)$$

$$= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$$

6. But there's a problem...
   We could keep adding $\frac{12\pi}{6}$ or 2π to our solutions to find equivalent angles $forever$, right? Because all we're doing is going around the unit circle again and again.
   How do we know when to stop?

   This is what our *domain* of $0 \leq x \leq 2\pi$ tells us.

   But notice that our domain tells us what values x should be between, whereas our equation currently is expressed in terms of $3x$ ...

$$\tan 3x = \pm\frac{1}{\sqrt{3}}$$

$$\therefore 3x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \left(\frac{\pi}{6} + \frac{12\pi}{6}\right), \left(\frac{5\pi}{6} + \frac{12\pi}{6}\right), \left(\frac{7\pi}{6} + \frac{12\pi}{6}\right), \left(\frac{11\pi}{6} + \frac{12\pi}{6}\right)$$

$$= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$$

We know that in order to express the equation in terms of x, we will need to *divide* all of our solutions by 3, which basically involves multiplying all of the denominators by 3, since for example:

$$\frac{\pi}{6} \div 3 = \frac{\pi}{6} \div \frac{3}{1} = \frac{\pi}{6} \times \frac{1}{3} = \frac{\pi}{18}$$

Since we can tell that when we divide both sides by 3 to get x by itself, all of our denominators will become 18, we can express the 2π in our domain $0 \leq x \leq 2\pi$ as a $fraction$ over 18, so that we can tell $when$ to stop adding 2π to our angles!

$$0 \leq x \leq 2\pi$$
$$0 \leq x \leq \frac{36\pi}{18}$$

$$\left\{ 2\pi = \frac{2\pi}{1} = \frac{2\pi \times 18}{1 \times 18} = \frac{2 \times 18 \times \pi}{18} = \frac{36\pi}{18} \right\}$$

Excellent! Now we know that we should stop adding 2π when our numerator goes above 36π :

$$\tan 3x = \pm\frac{1}{\sqrt{3}}$$

$$\therefore 3x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}, \frac{25\pi}{6}, \frac{29\pi}{6}, \frac{31\pi}{6}, \frac{35\pi}{6}$$

$+2\pi$ or $\frac{12\pi}{6}$

$+2\pi$ or $\frac{12\pi}{6}$

$$\star \quad \frac{25\pi}{6} + \frac{12\pi}{6} = \frac{37\pi}{6} \quad \times$$

out of our domain!

$$0 \leq x \leq \frac{36\pi}{18}$$

So, our solutions should stop at:

$$3x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}, \frac{25\pi}{6}, \frac{29\pi}{6}, \frac{31\pi}{6}, \frac{35\pi}{6}$$

Now we can divide by 3 on both sides to get x by itself, and we're done!

$$\therefore x = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{7\pi}{18}, \frac{11\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \frac{19\pi}{18}, \frac{23\pi}{18}, \frac{25\pi}{18}, \frac{29\pi}{18}, \frac{31\pi}{18}, \frac{35\pi}{18}$$

$$\left\{ 0 \leq x \leq \frac{36\pi}{18} \right\}$$

Exercise 1

Try a similar problem for yourself!

PROBLEM

Solve the following equation for x on $0 \leq x \leq 2\pi$:

$$\tan^2 2x = 1$$